Int J Theor Phys (2010) 49: 2971–2990
DOI 10.1007/s10773-010-0477-0

Quantum Experimental Data in Psychology and Economics

Diederik Aerts · Bart D'Hooghe · Emmanuel Haven

Received: 19 April 2010 / Accepted: 19 August 2010 / Published online: 23 September 2010

Abstract We prove a theorem which shows that a collection of experimental data of probabilistic weights related to decisions with respect to situations and their disjunction cannot be modeled within a classical probabilistic weight structure in case the experimental data contain the effect referred to as the 'disjunction effect' in psychology. We identify different experimental situations in psychology, more specifically in concept theory and in decision theory, and in economics (namely situations where Savage's Sure-Thing Principle is violated) where the disjunction effect appears and we point out the common nature of the effect. We analyze how our theorem constitutes a no-go theorem for classical probabilistic weight structures for common experimental data when the disjunction effect is affecting the values of these data. We put forward a simple geometric criterion that reveals the non classicality of the considered probabilistic weights and we illustrate our geometrical criterion by means of experimentally measured membership weights of items with respect to pairs of concepts and their disjunctions. The violation of the classical probabilistic weight structure is very analogous to the violation of the well-known Bell inequalities studied in quantum mechanics. The no-go theorem we prove in the present article with respect to the collection of experimental data we consider has a status analogous to the well known no-go theorems for hidden variable theories in quantum mechanics with respect to experimental data obtained in quantum laboratories. Our analysis puts forward a strong argument in favor of the validity of using the quantum formalism for modeling the considered psychological experimental data as considered in this paper.

Keywords Bell inequalities · Polytopes · Disjunction effect

D. Aerts · B. D'Hooghe
Leo Apostel Center for Interdisciplinary Studies, Brussels Free University, Brussels, Belgium

D. Aerts
e-mail: diraerts@vub.ac.be

B. D'Hooghe
e-mail: bdhooghe@vub.ac.be

E. Haven (✉)
School of Management, University of Leicester, Leicester, UK
e-mail: e.haven@le.ac.uk

1 Introduction

There exists an intensive ongoing research activity focusing on the use of the mathematical formalism of quantum mechanics to model situations in cognition and economics [1–4]. Our group at the Leo Apostel Center in Brussels has played a role in the initiation of this research domain [5–14], and is still actively engaged in it [15–22].

In the present article we make use of insights and techniques developed in the foundations of quantum mechanics to investigate whether a specific collection of experimental data can be modeled by means of a classical theory, or whether a more general theory is needed, eventually a quantum theory. For decades intensive research has been conducted with a focus on this very question, since physicists wanted to know whether quantum mechanics itself could be substituted by a classical theory. This body of research is traditionally referred to as 'the hidden variable problem of quantum mechanics', because indeed such a classical theory giving rise to the same predictions as quantum mechanics would be a theory containing 'hidden variables', to account for classical determinism on a hidden level. The presence of quantum-type probabilities would occur as a consequence of the lack of knowledge of hidden variables (which account for classical determinism on a hidden not necessarily manifest level). Physicists had already encountered such a situation before, namely classical statistical mechanics is a hidden variable theory for thermodynamics, i.e. the positions and velocities of the molecules of a given substance are hidden variables when the thermodynamic description level of the substance is the manifest level [23–40, 42–47]. John von Neumann proved the first no-go theorem which precludes the existence of hidden variables for quantum mechanics [23]. The famous Einstein-Podolsky-Rosen paradox proposal [24] was a next mile stone event with respect to the hidden variable question in quantum mechanics. Critical investigations of both von Neumann's no-go theorem and the EPR paradox were performed by Bell [28, 30]. Also an effective hidden variable theory was elaborated, nowadays called 'Bohm's theory', [29], followed by elaborations of von Neumann's theorem, i.e. further investigations from a structural perspective [26, 27, 31, 36, 38, 42–45, 47], and extensive discussions about several aspects of the problem [32–34, 46]. In the seventies the experimentalist became interested, and this led to new developments, e.g. the augmented understanding of notions such as locality, separability, etc.... But most of all, quantum mechanics was now confirmed as a robust physical theory, even when scrutinized under all types of aspects where failure could be expected in a plausible way [35, 37, 39, 40]. In the eighties, it was shown, step by step, that by focusing on the mathematical structure of the probability model used to model experimental data, it was possible to distinguish between data that is 'quantum' (more correctly 'non-classical', in the sense of not allowing a modeling within a classical Kolmogorovian probability model [41]), and data that is classical (hence can be modeled within a Kolmogorovian probability model) [38, 42–45, 47]. One of the aspects of this hidden variable research, which from the foundations of quantum mechanics point of view is definitely of more universal importance and value, is that the results with respect to the characterization of a set of experimental data, i.e. whether this data can be modeled within a classical theory or not, does not depend on whether this data is obtained from measurements in a physics laboratory. For sets of data whether obtained from experiments in psychology or economics (or in any other domain of science) the same analysis can be made, and the same techniques of characterization of the data can be employed.

We have already investigated in this way data that was gathered by experiments measuring membership weights of an item with respect to two concepts and the conjunction of these two concepts [48]. These experimental data provide experimental evidence for a quantum structure in cognition [17]. The deviation of what a classical probability theory would provide in modeling these experimental data was called 'overextension' in concept research circles [48]. Many experiments performed by different concept researchers have been able to measure the presence of 'overextension' for the conjunction of concepts [49–54], such that the 'deviation from classicality' is experimentally well documented and abundant. There is a correspondence between 'overextension' for typicality and membership weight values for the conjunction of concepts, and what in decision theory is referred to as 'the conjunction fallacy' [55, 56]. In the present article we want to concentrate on the 'disjunction', and 'how deviations from classicality appear when the disjunction is at play'. The experimental data that we consider as our element of study is the result of measurements of membership weights of items with respect to pairs of concepts and their disjunction [57]. Conjunction deviations from classicality in concept theories relate to the conjunction fallacy in decision theory. In analogy to the latter, there is the well studied disjunction effect in decision theory which corresponds to these disjunction deviations in concept theories [58–69]. Historically it was first in economics that the deviating effects, which psychologists later indicated with the disjunction effect and conjunction fallacy, were observed and identified as deviations of rational thought. Maurice Allais already in 1953 [70] and Daniel Ellsberg in 1961 [71] put forward specific situations of decision making in economics entailing such class of effects. More specifically, the Ellsberg paradox and the Allais paradox refer respectively, to a violation of Savage's Sure-Thing Principle [72], a fundamental hypothesis of subjective expected utility theory, and a violation of the independence axiom [73] which is a fundamental hypothesis of objective expected utility theory. Although we have used experimental data from psychology experiments for our no-go theorem, we could as well have gathered data related to examples from economics, which is the reason that we have chosen to explicitly mention also 'economics' in our title.

Before putting forward a simple criterion and also a geometric interpretation of it in the next section, we would like to mention that the disjunction effect in decision theory has been modeled quantum mechanically by several authors [74–76]. The disjunction effect, as it appears for the membership weights of items with respect to pairs of concepts and their disjunction, was modeled explicitly by the quantum mechanical formalism in our Brussels group [14, 15, 19]. The result of the present article, namely that the disjunction effect cannot be modeled classically, or more specifically that we prove a no-go theorem with respect to the classical weight structure developed within measure theory for a collection of experimental data of membership weights, supports the quantum models that have been put forward for it. This is the reason that we allowed ourselves to write 'quantum experimental data' in the title, although strictly speaking we should write 'non classical experimental data'. We prefer 'quantum' instead of 'non classical' because 'non classical' has only known meaning amongst those physicists and mathematicians who focus their research on quantum structures, and we want our title to speak to a broader audience, since the data we consider are gathered in a psychology research environment. This being said, we want to mention that we use 'quantum' only as referring to 'the structural aspects of quantum theory'. Hence, with 'quantum experimental data' we 'do not' mean that there is some hidden micro mechanical mechanism producing these data. Instead we mean that the structure of possible theories being able to model these data will show to be non classical, and that there is a strong plausibility for such theories to also contain definite aspects of quantum structure.

2 Classical and non Classical Membership Weights for Concepts

The disjunction experiments we want to focus on in the present article were performed with the aim of measuring deviations for membership weights of items with respect to concepts from how one would expect such membership weights to behave classically [57]. For example, the concepts *Home Furnishings* and *Furniture* and their disjunction '*Home Furnishings or Furniture*' are considered. With respect to this pair, the item *Ashtray* is considered. Subjects rated the membership weight of *Ashtray* for the concept *Home Furnishings* as 0.7 and the membership weight of the item *Ashtray* for the concept *Furniture* as 0.3. However, the membership weight of *Ashtray* with respect to the disjunction '*Home Furnishings or Furniture*' was rated as only 0.25, i.e. less than either one of the weights assigned for both concepts separately. This means that subjects found *Ashtray* to be 'less strongly a member of the disjunction '*Home Furnishings or Furniture*' than they found it to be a member of the concept *Home Furnishings* alone or a member of the concept *Furniture* alone'. If one thinks intuitively about the 'logical' meaning of a disjunction, then this is an unexpected result. Indeed, if somebody finds that *Ashtray* belongs to *Home Furnishings*, they would be expected to also believe that *Ashtray* belongs to '*Home Furnishings or Furniture*'. The same holds for *Ashtray* and *Furniture*. Hampton called this deviation (this relative to what one would expect according to a standard classical interpretation of the disjunction) 'underextension' [57].

A typical experiment testing the effect described above proceeds as follows. The tested subjects are asked to choose a number from the following set: $\{-3, -2, -1, 0, +1, +2, +3\}$, where the positive numbers $+1$, $+2$ or $+3$ mean that they consider 'the item to be a member of the concept' and the typicality of the membership increases with an increasing number. Hence $+3$ means that the subject who attributes this number considers the item to be a very typical member, and $+1$ means that he or she considers the item to be a not so typical member. The negative numbers indicate non-membership, again in increasing order, i.e. -3 indicates strong non-membership, and -1 represents weak non-membership. Choosing 0 indicates the subject is indecisive about the membership or non-membership of the item. Subjects were asked to repeat the procedure for all the items and concepts considered. Membership weights were then calculated by dividing the number of positive ratings by the number of non-zero ratings.

Consider again the case of *Ashtray* as an item and its membership with respect to the concepts *Home Furnishings* and *Furniture* and their disjunction. As the experiments are conceived, each individual subject will decide for *Ashtray* whether it is a member or not a member of respectively *Home Furnishings*, *Furniture* and '*Home Furnishings or Furniture*'. Suppose that there are n subjects participating in the experiment. There is a way to express what we mean intuitively by 'classical behavior'. Indeed, what we would 'not' like to happen is that a subject, decides *Ashtray* to be a member of *Home Furnishings*, but not a member of *Home Furnishings or Furniture*. If a subject would make such type of decision, then this would be in direct conflict with the meaning of the disjunction. However, in the case of *Ashtray*, since $0.7 \times n$ subjects have decided that *Ashtray* is a member of *Home Furnishings* and only $0.25 \times n$ subjects have decided it to be a member of '*Home Furnishings or Furniture*', this means that at least $0.45 \times n$ subjects have taken this decision in direct conflict with the meaning of the disjunction. In case $n = 100$, this means 45 subjects have done so.

Suppose we introduce the following notation, and indicate with A_1 the first considered concept, hence *Home Furnishings*, and with $\mu(A_1)$ the membership weight of item X, hence *Ashtray*, with respect to A_1. This means that for our example we have $\mu(A_1) = 0.7$. With A_2 we denote the second considered concept, hence *Furniture*, and with $\mu(A_2)$ the membership weight of item X, hence *Ashtray*, with respect to A_2. This means that for our example we have $\mu(A_2) = 0.3$. With 'A_1 or A_2' we denote the disjunction of both concepts A_1 and A_2, hence '*Home Furnishings or Furniture*', and with $\mu(A_1 \text{ or } A_2)$ the membership weight of item X, hence *Ashtray*, with respect to 'A_1 or A_2'. This means that in our example we have $\mu(A_1 \text{ or } A_2) = 0.25$.

We can easily see that the non classical effect we analyzed above cannot happen in case the following two inequalities are satisfied

$$\mu(A_1) \leq \mu(A_1 \text{ or } A_2) \quad \mu(A_2) \leq \mu(A_1 \text{ or } A_2) \tag{1}$$

and we observe indeed that both inequalities are violated for our example of *Ashtray* with respect to *Home Furnishings* and *Furniture*.

There is another issue which we do not want to happen, and this one is somewhat more subtle. To illustrate it, we consider another example of the experiments, namely the item *Olive*, with respect to the pair of concepts *Fruits* and *Vegetables* and their disjunction '*Fruits or Vegetables*'. The respective membership weights were measured to be $\mu(A_1) = 0.5$, $\mu(A_2) = 0.1$ and $\mu(A_1 \text{ or } A_2) = 0.8$. Obviously inequalities (1) are both satisfied for this example. Let us suppose again that there are n subjects participating in the experiment. Then $0.5 \times n$ subjects have decided that *Olive* is a member of *Fruits*, and $0.1 \times n$ subjects have decided that *Olive* is a member of *Vegetables*, while $0.8 \times n$ subjects have decided that *Olive* is a member of '*Fruits or Vegetables*'. However, at maximum $0.5 \times n + 0.1 \times n = 0.6 \times n$ subjects have decided that *Olive* is a member of *Fruits* 'or' is a member of *Vegetables*. This means that a minimum of $0.4 \times n$ subjects have decided that *Olive* is neither a member of *Fruits* nor a member of *Vegetables*. But $0.8 \times n$ subjects have decided that *Olive* is a member of '*Fruits or Vegetables*'. This means that a minimum of $0.2 \times n$ subjects have decided that *Olive* 'is not' a member of *Fruits*, and also 'is not' a member of *Vegetables*, but 'is' a member of '*Fruits or Vegetables*'. The decision made by these $0.2 \times n$ subjects, hence 20 in case $n = 100$, goes directly against the meaning of the disjunction. An item becoming a member of the disjunction while it is not a member of both pairs is completely non classical. We can easily see that this second type of non classicality cannot happen in case the following inequality holds

$$\mu(A_1 \text{ or } A_2) \leq \mu(A_1) + \mu(A_2) \tag{2}$$

and indeed this inequality is violated by the example of *Olive* with respect to *Fruits* and *Vegetables*. One of the authors has derived in earlier work the three inequalities (1) and (2) as a consequence of a different type of requirement, namely the requirement that the membership weights are in their most general form representations of mathematical normed measures (see Sect. 1.4, Theorem 4 and Appendix B of [15], and Theorem 4.1 of the present article). The items that deviated from classicality by violating one or both of the inequalities (1) were called Δ-type non classical items. The items that deviated from classicality by violating inequality (2) were called k-type non classical items. An explicit quantum model was constructed for both types of non classical items [15]. In the present paper we consider the general situation of n concepts and disjunctions of pairs of these n concepts, and we introduce the corresponding definition for classicality. We will additionally derive a simple geometrical criterion to verify whether the membership weights of an item with respect to a set of concepts and disjunctions of pairs of them can be modeled classically or not. Tables 1–8 represent the items and pairs of concepts tested by Hampton [57] which we will use as experimental data to illustrate the analysis put forward in the present article.

We consider n concepts $A_1, A_2, \ldots, A_n$ and membership weights $\mu(A_i)$ of an item X with respect to each concept A_i, and also membership weights $\mu(A_i \text{ or } A_j)$ of this item X with respect to the disjunction of concepts A_i and A_j. It is not necessary that membership weights of the item X are determined with respect to each one of the possible pairs of concepts. Hence, to describe this situation formally, we consider a set S of pairs of indices $S \subseteq \{(i,j) \mid i < j; i,j = 1,2,\ldots,n\}$ corresponding to those pairs of concepts for which the membership weights of the item X have been measured with respect to the disjunction of these pairs. Hence, the following set of membership weights have been experimentally determined

$$p_i = \mu(A_i) \quad i = 1, 2, \ldots, n; \qquad p_{i\vee j} = \mu(A_i \text{ or } A_j) \quad (i, j) \in S \tag{3}$$

Definition 2.1 (Classical Disjunction Data) We say that the set of membership weights of an item X with respect to concepts is a 'classical disjunctive set of membership weights' if it has a normed measure representation. Hence if there exists a normed measure space $(\Omega, \sigma(\Omega), P)$ with $E_{A_1}, E_{A_2}, \ldots, E_{A_n} \in \sigma(\Omega)$ elements of the event algebra, such that

$$p_i = P(E_{A_i}) \quad i = 1, 2, \ldots, n; \qquad p_{i\vee j} = P(E_{A_i} \cup E_{A_j}) \quad (i, j) \in S \tag{4}$$

A normed measure P is a function defined on a σ-algebra $\sigma(\Omega)$ over a set Ω which takes values in the interval [0, 1] such that the following properties are satisfied: (i) The empty set has measure zero, i.e. $P(\emptyset) = 0$; (ii) Countable additivity or σ-additivity: if E_1, E_2, E_3, ... is a countable sequence of pairwise disjoint sets in $\sigma(\Omega)$, the measure of the union of all the E_i is equal to the sum of the measures of each E_i, i.e. $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$; (iii) The total measure is one, i.e. $P(\Omega) = 1$. The triple $(\Omega, \sigma(\Omega), P)$ is called a normed measure space, and the members of $\sigma(\Omega)$ are called measurable sets. A σ-algebra over a set Ω is a nonempty collection $\sigma(\Omega)$ of subsets of Ω that is closed under complementation and countable unions of its members. Measure spaces are the most general structures devised by mathematicians and physicists to represent weights.

3 Geometrical Characterization of Membership Weights

We now develop the geometric language that makes it possible to verify the existence of a normed measure representation for a set of weights. For this purpose we introduce the 'classical disjunction polytope' $d_c(n, S)$ in the following way. We construct an $n + |S|$ dimensional 'classical disjunction vector'

$$\vec{p} = \big(p_1, p_2, \ldots, p_n, \ldots, p_{i\vee j}, \ldots\big)$$

where $|S|$ is the cardinality of S. We consider the linear space $R(n, S) \cong \mathbb{R}^{n+|S|}$ consisting of all real vectors of this type. Next, let $\varepsilon \in \{0, 1\}^n$ be an arbitrary n-dimensional vector consisting of 0 and 1's. For each ε we construct the classical disjunction vector $\vec{v}^{\varepsilon} \in R\,(n, S)$ by putting:

$$\begin{aligned} v_i^{\varepsilon} &= \varepsilon_i & i &= 1, \ldots, n \\ v_{ij}^{\varepsilon} &= \max(\varepsilon_i, \varepsilon_j) = \varepsilon_i + \varepsilon_j - \varepsilon_i\varepsilon_j & (i, j) &\in S \end{aligned}$$

The set of convex linear combinations of $\vec{v}^{\varepsilon}$ we call the 'classical disjunction polytope' $d_c(n, S)$:

$$d_c(n, S) = \left\{ \vec{w} \in R(n, S) \;\middle|\; \vec{w} = \sum_{\varepsilon\in\{0,1\}^n} \lambda_{\varepsilon}\vec{v}^{\varepsilon};\, \lambda_{\varepsilon} \geq 0;\, \sum_{\varepsilon\in\{0,1\}^n} \lambda_{\varepsilon} = 1 \right\}$$

We prove now the following theorem

Theorem 3.1 *The set of weights*

$$p_i = \mu(A_i) \quad i = 1, 2, \ldots, n; \qquad p_{i\vee j} = \mu(A_i \text{ } or \text{ } A_j) \quad (i, j) \in S$$

admits a normed measure space, and hence is a classical disjunction set of membership weights, if and only if its disjunction vector $\vec{p}$ belongs to the classical disjunction polytope $d_c(n, S)$.

Proof Suppose that (3) is a classical disjunction set of weights, and hence we have a normed measure space $(\Omega, \sigma(\Omega), P)$ and events $E_{A_i} \in \sigma(\Omega)$ such that (4) are satisfied. Let us show that in this case $\vec{p} \in d_c(n, S)$. For an arbitrary subset $X \subset \Omega$ we define $X^1 = X$ and $X^0 = \Omega \backslash X$. Consider $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$ and define $A(\epsilon) = \bigcap_\epsilon A_i^{\epsilon_i}$. Then we have that $A(\epsilon) \cap A(\epsilon') = \emptyset$ for $\epsilon \neq \epsilon'$, $\bigcup_\epsilon A(\epsilon) = \Omega$, and $\bigcup_{\epsilon, \epsilon_j = 1} A(\epsilon) = A_j$. We put now $\lambda_\epsilon = P(A(\epsilon))$. Then we have $\lambda_\epsilon \geq 0$ and $\sum_\epsilon \lambda_\epsilon = 1$, and $p_i = P(A_i) = \sum_{\epsilon, \epsilon_i = 1} \lambda_\epsilon = \sum_\epsilon \lambda_\epsilon \epsilon_i$. We also have $p_{i\vee j} = P(A_i \cup A_j) = \sum_{\epsilon, \max(\epsilon_i, \epsilon_j) = 1} \lambda_\epsilon = \sum_\epsilon \lambda_\epsilon (\epsilon_i + \epsilon_j - \epsilon_i \epsilon_j)$. This means that $\vec{p} = \sum_\epsilon \lambda_\epsilon v^\epsilon$, which shows that $\vec{p} \in d_c(n, S)$. Conversely, suppose that $\vec{p} \in d_c(n, S)$. Then there exist numbers $\lambda_\epsilon \geq 0$ such that $\sum_\epsilon \lambda_\epsilon = 1$ and $\vec{p} = \sum_\epsilon \lambda_\epsilon v^\epsilon$. We define $\Omega = \{0, 1\}^n$ and $\sigma(\Omega)$ the power set of Ω. For $X \subset \Omega$ we define then $P(X) = \sum_{\epsilon \in X} \lambda_\epsilon$. Then we choose $A_i = \{\epsilon, \epsilon_i = 1\}$ which gives that $P(A_i) = \sum_\epsilon \lambda_\epsilon \epsilon_i = \sum_\epsilon \lambda_\epsilon v_i^\epsilon = p_i$ and $P(A_i \cup A_j) = \sum_\epsilon \lambda_\epsilon \left(\epsilon_i + \epsilon_j - \epsilon_i \epsilon_j\right) = \sum_\epsilon \lambda_\epsilon v_{ij}^\epsilon = p_{i\vee j}$. This shows that we have a classical disjunction set of weights. □

As one may notice, these results are very similar to those of Pitowsky for classical conjunction polytopes $c(n, S)$ [47]. However, the Pitowsky correlation polytope and the classical disjunction polytope have different sets of vertices. Furthermore, the interpretation of the $|S|$ components is completely different, namely representing conjunction data p_{ij} and disjunction data $p_{i\vee j}$ respectively. In general, the existence of a classical disjunctive representation does not necessarily imply the existence of a classical conjunctive representation, and vice versa. Therefore, in order to fully grasp classicality by these geometric means, the natural next step is to combine the theoretical results for conjunction (Pitowsky) and disjunction polytopes (developed here and in [15]) by introducing a 'classical connective polytope'.

Again, let $\varepsilon \in \{0, 1\}^n$ be an arbitrary n-dimensional vector consisting of 0 and 1's. For each ε we construct the classical connective vector $\vec{w}^\varepsilon \in \mathbb{R}^{n+|S|+|S'|}$ by putting:

$$\begin{aligned} w_i^\varepsilon &= \varepsilon_i && i = 1, \ldots, n \\ w_{ij}^\varepsilon &= \varepsilon_i \varepsilon_j = \min(\varepsilon_i, \varepsilon_j) && (i, j) \in S \\ w_{k\vee l}^\varepsilon &= \varepsilon_k + \varepsilon_l - \varepsilon_k \varepsilon_l = \max(\varepsilon_k, \varepsilon_l) && (k, l) \in S' \end{aligned}$$

The set of convex linear combinations of $\vec{w}^\varepsilon$ we call the 'classical connective polytope' $k(n, S, S')$:

$$k(n, S, S') = \left\{ \vec{f} \in \mathbb{R}^{n+|S|+|S'|} \,\middle|\, \vec{f} = \sum_{\epsilon \in \{0,1\}^n} \lambda_\epsilon \vec{w}^\epsilon;\ \lambda_\epsilon \geq 0;\ \sum_{\epsilon \in \{0,1\}^n} \lambda_\epsilon = 1 \right\} \tag{5}$$

Theorem 3.2 *The set of weights*

$$p_i = \mu(A_i) \quad i = 1, 2, \ldots, n; \qquad p_{ij} = \mu(A_i \text{ and } A_j) \quad (i, j) \in S;$$

$$p_{i\vee j} = \mu(A_i \text{ or } A_j) \quad (i, j) \in S'$$

admits a normed measure space, and hence is a classical set of membership weights, if and only if its connective vector $\vec{p}$ belongs to the classical connective polytope $k(n, S, S')$.

Proof Follows from the theorems for conjunction and disjunction classicality. □

4 A Simple Case: Disjunction Effect for 2 Concepts

One of the authors studied the disjunction effect for the case of two concepts and their disjunction [15]. We recall Theorem 4 of [15].

Theorem 4.1 *The membership weights $\mu(A)$, $\mu(B)$ and $\mu(A \text{ or } B)$ of an item X with respect to concepts A and B and their disjunction 'A or B' are classical disjunction data if and only if they satisfy the following 'classical disjunction' inequalities*:

$$(i)\ 0 \le \mu(A) \le \mu(A \text{ or } B) \le 1 \qquad (ii)\ 0 \le \mu(B) \le \mu(A \text{ or } B) \le 1$$
$$(iii)\ 0 \le \mu(A) + \mu(B) - \mu(A \text{ or } B) \tag{6}$$

Proof See [15]. □

In the case of two concepts A_1, A_2 and their disjunction 'A_1 or A_2' the set of indices is $S = \{(1,2)\}$ and the classical disjunction polytope $d_c(n, S)$ is contained in the $2 + |S| = 3$ dimensional Euclidean space, i.e. $R(2, \{1,2\}) = \mathbb{R}^3$. Furthermore we have four vectors $\epsilon \in \{0,1\}^n$, namely $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$, and hence the four vectors $\vec{v}^\epsilon \in \mathbb{R}^3$ which are the following

$$\vec{v}^{(0,0)} = (0,0,0) \qquad \vec{v}^{(1,0)} = (1,0,1) \qquad \vec{v}^{(0,1)} = (0,1,1) \qquad \vec{v}^{(1,1)} = (1,1,1) \tag{7}$$

This means that the correlation polytope $d_c(n, S)$ is the convex region spanned by the convex combinations of the vectors $(0,0,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$, and the disjunction vector is given by $\vec{p} = (\mu(A_1), \mu(A_2), \mu(A_1 \text{ or } A_2))$. It is well-known that every polytope admits two dual descriptions: one in terms of convex combinations of its vertices, and one in terms of the inequalities that define its boundaries. Following [15], the inequalities defining the boundaries for the polytope $d_c(2, \{(1,2)\})$ are given by:

$$0 \le p_1 \le p_{1\vee 2} \le 1, \tag{8}$$
$$0 \le p_2 \le p_{1\vee 2} \le 1, \tag{9}$$
$$0 \le p_1 + p_2 - p_{1\vee 2} \le 1 \tag{10}$$

We observe that the last inequality $p_1 + p_2 - p_{1\vee 2} \le 1$ follows easily because from $p_1 \le p_{1\vee 2}$ and $p_2 \le p_{1\vee 2}$ follows that $p_1 + p_2 - p_{1\vee 2} \le p_{1\vee 2} \le 1$ (again because of (8)).

Theorem 4.2 *The classical disjunction inequalities formulated in Theorem 4.1 are satisfied if and only if $\vec{p} \in d_c(2, \{(1,2)\})$.*

Proof Let us notice that

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_{1\vee 2} \end{pmatrix} = (1 - p_{1\vee 2}) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + (p_{1\vee 2} - p_1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (p_{1\vee 2} - p_2) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$+ (p_1 + p_2 - p_{1\vee 2}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= a \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + d \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence if the classical disjunction inequalities formulated in Theorem 4.1 are satisfied, then it is easy to check that $\vec{p} \in d_c(2, \{(1,2)\})$. Vice versa, let $\vec{p} \in d_c(2, \{(1,2)\})$. Rewriting $\vec{p}$ as above, and putting condition $0 \le a, b, c, d \le 1$, then the classical disjunction inequalities of Theorem 4.1 follow:

$$0 \le 1 - p_{1\vee 2} \le 1$$
$$0 \le p_{1\vee 2} - p_1 \le 1$$
$$0 \le p_{1\vee 2} - p_2 \le 1$$
$$0 \le p_1 + p_2 - p_{1\vee 2} \le 1$$

The last inequality *is* condition (10), while $0 \le 1 - p_{1\vee 2} \Rightarrow p_{1\vee 2} \le 1$. Also $0 \le p_{1\vee 2} - p_1 \Rightarrow p_1 \le p_{1\vee 2}$. Also, $0 \le p_1 + p_2 - p_{1\vee 2}$ implies that $p_{1\vee 2} - p_2 \le p_1$ and since $0 \le p_{1\vee 2} - p_2$ follows that $0 \le p_{1\vee 2} - p_2 \le p_1$ so $0 \le p_1$. Putting these together, we obtain then $0 \le p_1 \le p_{1\vee 2} \le 1$. Similarly, we can prove $0 \le p_2 \le p_{1\vee 2} \le 1$. □

For example, let us consider the experimental data in Table 1. In Fig. 1 we have represented the disjunction vectors formed by the membership weights of the different items to be found in Table 1 with respect to the pairs of concepts *Home Furnishings* and *Furniture* and their disjunction '*Home Furnishings or Furniture*', and also the disjunction polytope. The classical items, hence with disjunction vector inside the polytope, are represented by a little open disk. They are *Desk*, *Bed*, *Rug*, *Wall-Hangings*, *Shelves*, *Sculpture*, *Bath Tub*, *Door Bell* and *Desk Chair*. The non classical items, hence with disjunction vector outside of the polytope, are represented by a little closed disk. They are *Lamp*, *Wall Mirror*, *Window Seat*, *Painting*, *Light Fixture*, *Mantelpiece*, *Refrigerator*, *Space Rack*, *Sink Unit*, *Waste Paper Basket*, *Kitchen Count*, *Bar*, *Hammock*, *Ashtray* and *Park Bench*. In a similar way, we have represented the data of Table 1 in Fig. 1, Table 2 in Fig. 2, Table 3 in Fig. 3, Table 4 in Fig. 4, Table 5 in Fig. 5, Table 6 in Fig. 6, Table 7 in Fig. 7, and Table 8 in Fig. 8.

Remark that if the experimental data turns out such that the item is a classical item, this does not mean that perhaps non classical effects are not at play also for this item. But the non classical effects might be such that they do not show up with these particular measurements. This aspect of the situation is analyzed in more detail in [15].

The inequalities that define the boundaries of polytope $d_c(n, S)$ are a variant of the well-known Bell inequalities [28, 47], studied in the foundations of quantum mechanics, but now put into the context of disjunctive connectives instead of conjunctive correlations. This means that the violation of these inequalities, such as it happens by the data corresponding to items for which the points lie outside the polytope, has from a probabilistic perspective an analogous meaning as the violation of Bell inequalities for the conjunction. Hence these violations may indicate the presence of quantum structures in the domain where the data is collected, which makes it plausible that a quantum model, such as for example the one proposed in [15], can be used to model the data.

As we have shown above, the classical disjunction polytope allows for one necessary and sufficient condition $\vec{p} \in d_c(n, S)$ which guarantees a classical Kolmogorovian model for the given set of probabilities to exist [41]. As illustrated here, this can be expressed

Table 1 The data corresponding to the pairs of concepts *Home Furnishing* and *Furniture* of experiment 2 in [57]. $\mu(A_1)$, $\mu(A_2)$ and $\mu(A_1 \text{ or } A_2)$ are respectively the measured membership weights with respect to the concepts A_1, A_2 and their disjunction A_1 or A_2. The non classical items are labeled by q and the classical items by c

		$\mu(A_1)$	$\mu(A_2)$	$\mu(A_1 \text{ or } A_2)$
$A_1 =$ *Home Furnishing*, $A_2 =$ *Furniture*				
Mantelpiece	q	0.8	0.4	0.75
Window Seat	q	0.9	0.9	0.8
Painting	q	0.9	0.5	0.85
Light Fixture	q	0.8	0.4	0.775
Kitchen Count	q	0.8	0.55	0.625
Bath Tub	c	0.5	0.7	0.75
Desk Chair	c	0.1	0.3	0.35
Shelves	c	1	0.4	1
Rug	c	0.9	0.6	0.95
Bed	c	1	1	1
Wall-Hangings	c	0.9	0.4	0.95
Space Rack	q	0.7	0.5	0.65
Ashtray	q	0.7	0.3	0.25
Bar	q	0.35	0.6	0.55
Lamp	q	1	0.7	0.9
Wall Mirror	q	1	0.6	0.95
Door Bell	c	0.5	0.1	0.55
Hammock	q	0.2	0.5	0.35
Desk	c	1	1	1
Refrigerator	q	0.9	0.7	0.575
Park Bench	q	0	0.3	0.05
Waste Paper Basket	q	1	0.5	0.6
Sculpture	c	0.8	0.4	0.8
Sink Unit	q	0.9	0.6	0.6

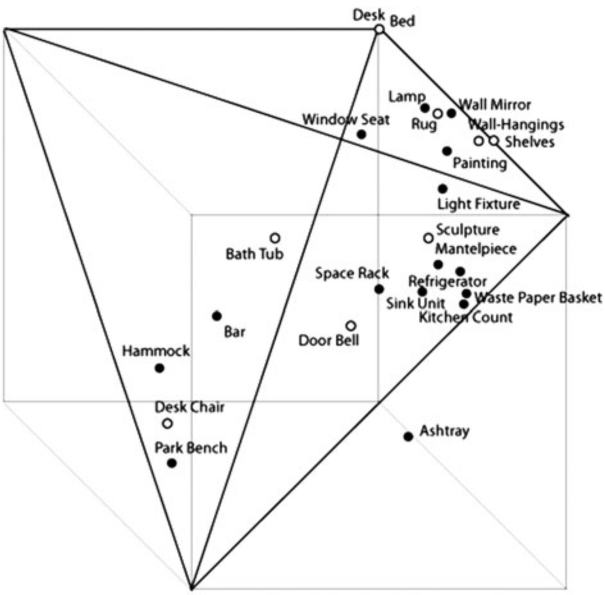

Fig. 1 The polytopes for the concepts *Home Furnishing* and *Furniture*. The classical items correspond to an open disk while the quantum ones to a full disk

Table 2 The data corresponding to the pairs of concepts *Spices* and *Herbs* of experiment 2 in [57]. $\mu(A_1)$, $\mu(A_2)$ and $\mu(A_1 \text{ or } A_2)$ are respectively the measured membership weights with respect to the concepts A_1, A_2 and their disjunction A_1 or A_2. The non classical items are labeled by q and the classical items by c

		$\mu(A_1)$	$\mu(A_2)$	$\mu(A_1 \text{ or } A_2)$
$A_1 =$ *Spices*, $A_2 =$ *Herbs*				
Molasses	*c*	0.4	0.05	0.425
Salt	*q*	0.75	0.1	0.6
Peppermint	*c*	0.45	0.6	0.6
Curry	*q*	0.9	0.4	0.75
Oregano	*q*	0.7	1	0.875
MSG	*q*	0.15	0.1	0.425
Chili Pepper	*q*	1	0.6	0.95
Mustard	*q*	1	0.8	0.85
Mint	*c*	1	0.8	0.925
Cinnamon	*c*	1	0.4	1
Parsley	*c*	0.5	0.9	0.95
Saccharin	*q*	0.1	0.01	0.15
Poppyseeds	*c*	0.4	0.4	0.4
Pepper	*c*	0.9	0.6	0.95
Turmeric	*q*	0.7	0.45	0.675
Sugar	*q*	0	0	0.2
Vinegar	*q*	0.1	0.01	0.35
Sesame Seeds	*c*	0.35	0.4	0.625
Lemon Juice	*q*	0.1	0.01	0.15
Chocolate	*c*	0	0	0
Horseradish	*q*	0.2	0.4	0.7
Vanilla	*q*	0.6	0	0.275
Chires	*q*	0.6	1	0.95
Root Ginger	*q*	0.7	0.15	0.675

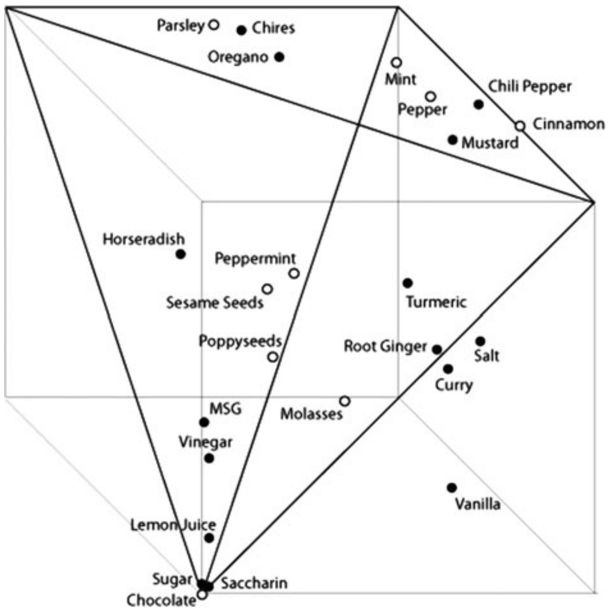

Fig. 2 The polytopes for the concepts *Spices* and *Herbs*. The classical items correspond to an open disk while the quantum ones to a full disk

Table 3 The data corresponding to the pairs of concepts *Hobbies* and *Games* of experiment 2 in [57]. $\mu(A_1)$, $\mu(A_2)$ and $\mu(A_1 \text{ or } A_2)$ are respectively the measured membership weights with respect to the concepts A_1, A_2 and their disjunction A_1 or A_2. The non classical items are labeled by q and the classical items by c

		$\mu(A_1)$	$\mu(A_2)$	$\mu(A_1 \text{ or } A_2)$
A = *Hobbies, B* = *Games*				
Gardening	*c*	1	0	1
Theatre-Going	*c*	1	0	1
Archery	*q*	1	0.9	0.95
Monopoly	*c*	0.7	1	1
Tennis	*c*	1	1	1
Bowling	*c*	1	1	1
Fishing	*c*	1	0.6	1
Washing Dishes	*q*	0.1	0	0.15
Eating Ice-Cream Cones	*q*	0.2	0	0.1
Camping	*q*	1	0.1	0.9
Skating	*q*	1	0.6	0.95
Judo	*q*	1	0.7	0.8
Guitar Playing	*c*	1	0	1
Autograph Hunting	*q*	1	0.2	0.9
Discus Throwing	*q*	1	0.75	0.7
Jogging	*q*	1	0.4	0.9
Keep Fit	*q*	1	0.3	0.95
Noughts	*q*	0.5	1	0.9
Karate	*q*	1	0.7	0.8
Bridge	*c*	1	1	1
Rock Climbing	*q*	1	0.2	0.95
Beer Drinking	*q*	0.8	0.2	0.575
Stamp Collecting	*c*	1	0.1	1
Wrestling	*q*	0.9	0.6	0.625

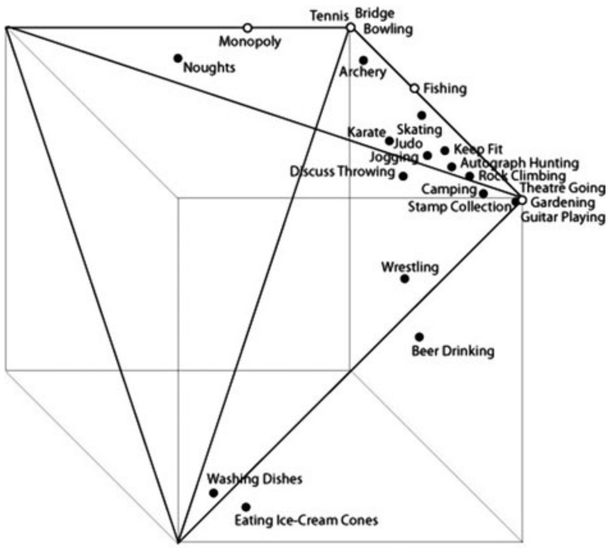

Fig. 3 The polytopes for the concepts *Hobbies* and *Games*. The classical items correspond to an open disk while the quantum ones to a full disk

Table 4 The data corresponding to the pairs of concepts *Instruments* and *Tools* of experiment 2 in [57]. $\mu(A_1)$, $\mu(A_2)$ and $\mu(A_1 \text{ or } A_2)$ are respectively the measured membership weights with respect to the concepts A_1, A_2 and their disjunction A_1 or A_2. The non classical items are labeled by q and the classical items by c

		$\mu(A_1)$	$\mu(A_2)$	$\mu(A_1 \text{ or } A_2)$
A = Instruments, B = Tools				
Broom	q	0.1	0.7	0.6
Magnetic Compass	c	0.9	0.5	1
Tuning Fork	c	0.9	0.6	1
Pen-Knife	q	0.65	1	0.95
Rubber Band	q	0.25	0.5	0.25
Stapler	c	0.85	0.8	0.85
Skate Board	q	0.1	0	0
Scissors	q	0.85	1	0.9
Pencil Eraser	q	0.4	0.7	0.45
Tin Opener	c	0.9	0.9	0.95
Bicycle Pump	q	1	0.9	0.7
Scalpel	q	0.8	1	0.925
Computer	q	0.6	0.8	0.6
Paper Clip	q	0.3	0.7	0.6
Paint Brush	c	0.65	0.9	0.95
Step Ladder	q	0.2	0.9	0.85
Door Key	q	0.3	0.1	0.95
Measuring Calipers	q	0.9	1	0.9
Toothbrush	c	0.4	0.4	0.5
Sellotape	q	0.1	0.2	0.325
Goggles	q	0.2	0.3	0.15
Spoon	q	0.65	0.9	0.7
Pliers	c	0.8	1	1
Meat Thermometer	c	0.75	0.8	0.9

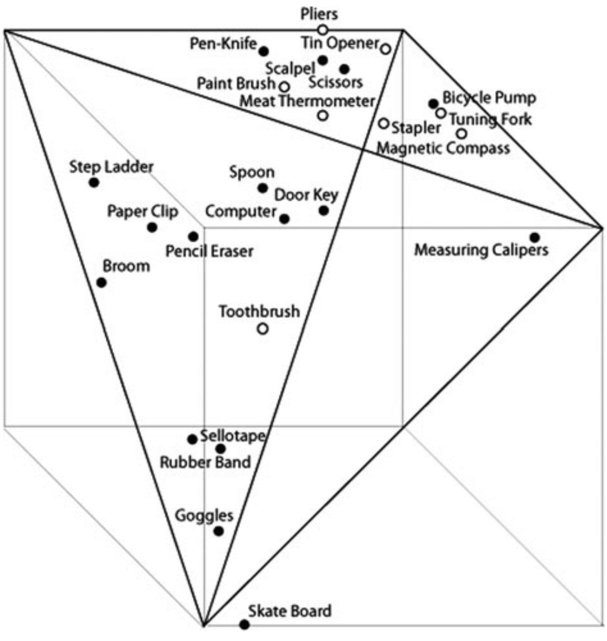

Fig. 4 The polytopes for the concepts *Instruments* and *Tools*. The classical items correspond to an open disk while the quantum ones to a full disk

Table 5 The data corresponding to the pairs of concepts *Pets* and *Farmyard Animals* of experiment 2 in [57]. $\mu(A_1)$, $\mu(A_2)$ and $\mu(A_1 \text{ or } A_2)$ are respectively the measured membership weights with respect to the concepts A_1, A_2 and their disjunction A_1 or A_2. The non classical items are labeled by q and the classical items by c

		$\mu(A_1)$	$\mu(A_2)$	$\mu(A_1 \text{ or } A_2)$
$A_1 = $ *Pets*, $A_2 = $ *Farmyard Animals*				
Goldfish	q	1	0	0.95
Robin	c	0.1	0.1	0.1
Blue-Tit	c	0.1	0.1	0.1
Collie Dog	c	1	0.7	1
Camel	q	0.4	0	0.1
Squirrel	q	0.2	0.1	0.1
GuideDog for the Blind	q	0.7	0	0.9
Spider	c	0.5	0.35	0.55
Homing Pig	q	0.9	0.1	0.8
Monkey	q	0.5	0	0.25
Circus Horse	q	0.4	0	0.3
Prize Bull	q	0.1	1	0.9
Rat	q	0.5	0.7	0.4
Badger	q	0	0.25	0.1
Siamese Cat	q	1	0.1	0.95
Race Horse	c	0.6	0.25	0.65
Fox	q	0.1	0.3	0.2
Donkey	q	0.5	0.9	0.7
Field Mouse	q	0.1	0.7	0.4
Ginger Tom-Cat	q	1	0.8	0.95
Husky in Sledream	q	0.4	0	0.425
Cart Horse	q	0.4	1	0.85
Chicken	q	0.3	1	0.95
Doberman Guard Dog	q	0.6	0.85	0.8

Fig. 5 The polytopes for the concepts *Pets* and *Farmyard Animals*. The classical items correspond to an open disk while the quantum ones to a full disk

Table 6 The data corresponding to the pairs of concepts *Fruits* and *Vegetables* of experiment 2 in [57]. $\mu(A_1)$, $\mu(A_2)$ and $\mu(A_1 \text{ or } A_2)$ are respectively the measured membership weights with respect to the concepts A_1, A_2 and their disjunction A_1 or A_2. The non classical items are labeled by q and the classical items by c

		$\mu(A_1)$	$\mu(A_2)$	$\mu(A_1 \text{ or } A_2)$
$A_1 = Fruits,\ A_2 = Vegetables$				
Apple	*c*	1	0	1
Parsley	*q*	0	0.2	0.45
Olive	*q*	0.5	0.1	0.8
Chili Pepper	*c*	0.05	0.5	0.5
Broccoli	*q*	0	0.8	1
Root Ginger	*q*	0	0.3	0.55
Pumpkin	*c*	0.7	0.8	0.925
Raisin	*q*	1	0	0.9
Acorn	*q*	0.35	0	0.4
Mustard	*q*	0	0.2	0.175
Rice	*q*	0	0.4	0.325
Tomato	*c*	0.7	0.7	1
Coconut	*q*	0.7	0	1
Mushroom	*q*	0	0.5	0.9
Wheat	*q*	0	0.1	0.2
Green Pepper	*c*	0.3	0.6	0.8
Watercress	*q*	0	0.6	0.8
Peanut	*c*	0.3	0.1	0.4
Black Pepper	*c*	0.15	0.2	0.225
Garlic	*q*	0.1	0.2	0.5
Yam	*c*	0.45	0.65	0.85
Elderberry	*q*	1	0	0.8
Almond	*q*	0.2	0.1	0.425
Lentils	*q*	0	0.6	0.525

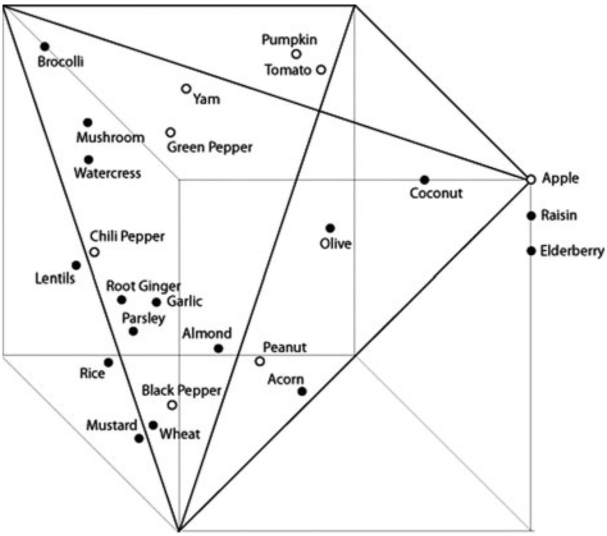

Fig. 6 The polytopes for the concepts *Fruits* and *Vegetables*. The classical items correspond to an open disk while the quantum ones to a full disk

Table 7 The data corresponding to the pairs of concepts *Sportswear* and *Sports Equipment* of experiment 2 in [57]. $\mu(A_1)$, $\mu(A_2)$ and $\mu(A_1 \text{ or } A_2)$ are respectively the measured membership weights with respect to the concepts A_1, A_2 and their disjunction A_1 or A_2. The non classical items are labeled by q and the classical items by c

		$\mu(A_1)$	$\mu(A_2)$	$\mu(A_1 \text{ or } A_2)$
A = Sportswear, B = Sports Equipment				
American Foot	*c*	1	1	1
Referee's Whistle	*q*	0.6	0.2	0.45
Circus Clowns	*q*	0	0	0.1
Backpack	*c*	0.6	0.5	0.6
Diving Mask	*q*	1	1	0.95
Frisbee	*q*	0.3	1	0.85
Sunglasses	*q*	0.4	0.2	0.1
Suntan Lotion	*q*	0	0	0.1
Gymnasium	*q*	0	0.9	0.825
Motorcycle Helmet	*q*	0.7	0.9	0.75
Rubber Flipper	*c*	1	1	1
Wrist Sweat	*q*	1	1	0.95
Golf Ball	*c*	0.1	1	1
Cheerleaders	*c*	0.3	0.4	0.45
Linesman's Flag	*q*	0.1	1	0.75
Underwater	*q*	1	0.65	0.6
Baseball Bat	*c*	0.2	1	1
Bathing Costume	*q*	1	0.8	0.8
Sailing Life Jacket	*c*	1	0.8	1
Ballet Shoes	*q*	0.7	0.6	0.6
Hoola Hoop	*q*	0.1	0.6	0.5
Running Shoes	*c*	1	1	1
Cricket Pitch	*q*	0	0.5	0.525
Tennis Racket	*c*	0.2	1	1

Fig. 7 The polytopes for the concepts *Sportswear* and *Sports Equipment*. The classical items correspond to an open disk while the quantum ones to a full disk

Table 8 The data corresponding to the pairs of concepts *Household Appliances* and *Kitchen Utensils* of experiment 2 in [57]. $\mu(A_1)$, $\mu(A_2)$ and $\mu(A_1 \text{ or } A_2)$ are respectively the measured membership weights with respect to the concepts A_1, A_2 and their disjunction A_1 or A_2. The non classical items are labeled by q and the classical items by c

		$\mu(A_1)$	$\mu(A_2)$	$\mu(A_1 \text{ or } A_2)$
A = Household Appliances, B = Kitchen Utensils				
Fork	*q*	0.7	1	0.95
Apron	*c*	0.3	0.4	0.5
Hat Stand	*q*	0.45	0	0.3
Freezer	*q*	1	0.6	0.95
Extractor Fan	*q*	1	0.4	0.9
Cake Tin	*c*	0.4	0.7	0.95
Carving Knife	*c*	0.7	1	1
Cooking Stove	*c*	1	0.5	1
Iron	*q*	1	0.3	0.95
Food Processor	*c*	1	1	1
Chopping Board	*q*	0.45	1	0.95
Television	*q*	0.95	0	0.85
Vacuum Cleaner	*c*	1	0	1
Rubbish Bin	*c*	0.5	0.5	0.8
Vegetable Rack	*c*	0.4	0.4	0.7
Broom	*c*	0.55	0.4	0.625
Rolling Pin	*c*	0.45	1	1
Table Mat	*q*	0.25	0.4	0.325
Whisk	*c*	1	1	1
Blender	*c*	1	1	1
Electric Toothbrush	*q*	0.8	0	0.55
Frying Pan	*q*	0.7	1	0.95
Toaster	*c*	1	1	1
Spatula	*c*	0.55	0.9	0.95

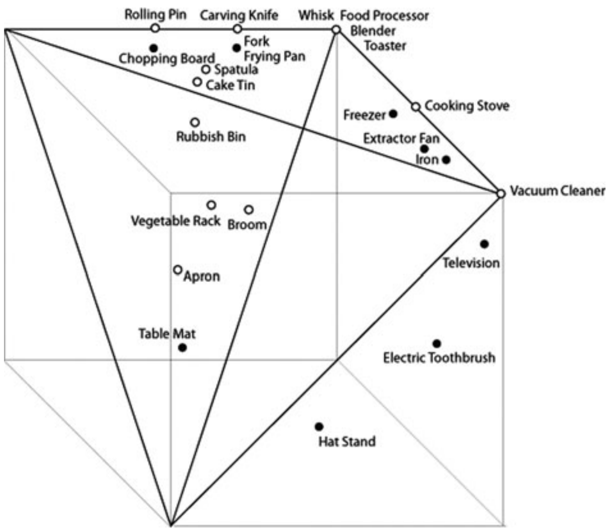

Fig. 8 The polytopes for the concepts *Household Appliances* and *Kitchen Utensils*. The classical items correspond to an open disk while the quantum ones to a full disk

by a set of Bell-type inequalities. However, as Pitowsky remarked [47], the number and complexity of the inequalities will grow so fast with n, that it would require exponentially many computation steps to derive them all. Anyway, already for the simplest (non-trivial) case $n = 2$ interesting inequalities can be derived by which the non classical nature of a set of statistical data can be demonstrated explicitly. Such data exists in various fields of science: of course in quantum mechanics, but also in cognition (concept) theory, decision theory and some paradoxical situations in economics, such as in the Allais and Ellsberg paradox situations [70, 71], notably situations which violate Savage's 'Sure-Thing principle' [72].

In summary we note that we have referred to our result as a no-go theorem. The theoretical part of theorems, such as the one of von Neumann [23], compare predictions of one theory, i.e. quantum mechanics, with modeling under specific requirements, i.e. classicality. This is not what we have done, which means that our result is analogous to the experimental part of the no-go theorem situation in quantum mechanics, namely where experimental results on quantum systems were shown to conflict with classical modeling.

Acknowledgements This work was supported by grants G.0405.08 and G.0234.08 of the Research Program of the Research Foundation-Flanders (FWO, Belgium).

References

1. Bruza, P.D., Gabora, L. (eds.): Special Issue: Quantum Cognition. J. Math. Psychol. **53**, 303–452 (2009)
2. Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Proceedings of the AAAI Spring Symposium on Quantum Interaction, March 26–28, 2007, Stanford University, SS-07-08. AAAI Press, Menlo Park (2007)
3. Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D., Coecke, B., Clark, S. (eds.): Proceedings of the Second Quantum Interaction Symposium, University of Oxford, March 26–28, 2008. College Publications, London (2008)
4. Bruza, P.D., Sofge, D., Lawless, W., van Rijsbergen, C.J., Klusch, M. (eds.): Proceedings of the Third International Symposium, QI 2009, Saarbrücken, Germany, March 25–27, 2009. Lecture Notes in Computer Science, vol. 5494. Springer, Berlin, Heidelberg (2009)
5. Aerts, D., Aerts, S.: Applications of quantum statistics in psychological studies of decision processes. Found. Sci. **1**, 85–97 (1994). Reprinted in: B.C. van Fraassen (ed.), Topics in the Foundation of Statistics, Springer, Dordrecht
6. Aerts, D., Broekaert, J., Smets, S.: The liar paradox in a quantum mechanical perspective. Found. Sci. **4**, 115–132 (1999)
7. Aerts, D., Broekaert, J., Smets, S.: A quantum structure description of the liar paradox. Int. J. Theor. Phys. **38**, 3231–3239 (1999)
8. Aerts, D., Aerts, S., Broekaert, J., Gabora, L.: The violation of Bell inequalities in the macroworld. Found. Phys. **30**, 1387–1414 (2000)
9. Gabora, L., Aerts, D.: Contextualizing concepts using a mathematical generalization of the quantum formalism. J. Exp. Theor. Artif. Intell. **14**, 327–358 (2002)
10. Aerts, D., Czachor, M.: Quantum aspects of semantic analysis and symbolic artificial intelligence. J. Phys. A **37**, L123–L132 (2004)
11. Aerts, D., Gabora, L.: A theory of concepts and their combinations I: the structure of the sets of contexts and properties. Kybernetes **34**, 167–191 (2005)
12. Aerts, D., Gabora, L.: A theory of concepts and their combinations II: a Hilbert space representation. Kybernetes **34**, 192–221 (2005)
13. Broekaert, J., Aerts, D., D'Hooghe, B.: The generalised Liar Paradox: a quantum model and interpretation. Found. Sci. **11**, 399–418 (2006)
14. Aerts, D.: General quantum modeling of combining concepts: a quantum field model in Fock space. Archive address and link: http://arxiv.org/abs/0705.1740 (2007)
15. Aerts, D.: Quantum structure in cognition. J. Math. Psychol. **53**, 314–348 (2009)
16. Aerts, D.: Quantum particles as conceptual entities: a possible explanatory framework for quantum theory. Found. Sci. **14**, 361–411 (2009)
17. Aerts, D., Aerts, S., Gabora, L.: Experimental evidence for quantum structure in cognition. In: Bruza, P.D., Sofge, D., Lawless, W., van Rijsbergen, C.J., Klusch, M. (eds.) Proceedings of QI 2009-Third International Symposium on Quantum Interaction. Lecture Notes in Computer Science, vol. 5494, pp. 59–70. Springer, Berlin, Heidelberg (2009)

18. Aerts, D., D'Hooghe, B.: Classical logical versus quantum conceptual thought: examples in economics, decision theory and concept theory. In: Bruza, P.D., Sofge, D., Lawless, W., van Rijsbergen, C.J., Klusch, M. (eds.) Proceedings of QI 2009-Third International Symposium on Quantum Interaction. Lecture Notes in Computer Science, vol. 5494, pp. 128–142 Springer, Berlin, Heidelberg (2009)
19. Aerts, D.: Quantum interference and superposition in cognition: Development of a theory for the disjunction of concepts. In: Aerts, D., D'Hooghe, B., Note, N. (eds.) Worldviews, Science and Us: Bridging Knowledge and Its Implications for Our Perspectives of the World. World Scientific, Singapore (2010)
20. Aerts, D.: Interpreting quantum particles as conceptual entities. Int. J. Theor. Phys. (2010). doi:10.1007/s10773-010-0440-0
21. Aerts, D., Broekaert, J., Gabora, L.: A case for applying an abstracted quantum formalism to cognition. New Ideas Psychol. (2010). doi:10.1016/j.newideapsych.2010.06.002
22. Aerts, D., D'Hooghe, B.: A quantum-conceptual explanation of violations of expected utility in economics. In: Aerts, D., D'Hooghe, B., Note, N. (eds.) Worldviews, Science and Us: Bridging Knowledge and Its Implications for Our Perspectives of the World. World Scientific, Singapore (2010)
23. von Neumann, J.: Mathematische Grundlagen der Quantenmechanik. Chapter IV.1,2. Springer, Berlin. (1932)
24. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777–780 (1935)
25. Bohr, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **48**, 696–702 (1935)
26. Gleason, A.M.: Measures on the closed subspaces of a Hilbert space. J. Math. Mech. **6**, 885–893 (1957)
27. Jauch, J., Piron, C.: Can hidden variables be excluded from quantum mechanics? Helv. Phys. Acta **36**, 827–837 (1963)
28. Bell, J.S.: On the Einstein Podolsky Rosen Paradox. Physics **1**, 195–200 (1964)
29. Bohm, D., Bub, J.: A proposed solution of the measurement problem in quantum mechanics by a hidden variable theory. Rev. Mod. Phys. **38**, 453–469 (1966)
30. Bell, J.S.: On the problem of hidden variables in quantum mechanics. Rev. Mod. Phys. **38**, 447–452 (1966)
31. Kochen, S., Specker, E.P.: The problem of hidden variables in quantum mechanics. J. Math. Mech. **17**, 59–87 (1967)
32. Jauch, J.M., Piron, C.: Hidden variables revisited. Rev. Mod. Phys. **40**, 228–229 (1968)
33. Gudder, S.P.: Hidden variables in quantum mechanics reconsidered. Rev. Mod. Phys. **40**, 229–231 (1968)
34. Bohm, D., Bub, J.: On hidden variables-A reply to comments by Jauch and Piron and by Gudder. Rev. Mod. Phys. **40**, 235–236 (1968)
35. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. **23**, 880–884 (1969)
36. Gudder, S.P.: On hidden variable theories. J. Math. Phys. **11**, 431–436 (1970)
37. Clauser, J.F., Horne, M.A.: Experimental consequences of objective local theories. Phys. Rev. D **10**, 526–535 (1974)
38. Accardi, L., Fedullo, A.: On the statistical meaning of complex numbers in quantum mechanics. Lett. Nuovo Cimento **34**, 161–172 (1982)
39. Aspect, A., Grangier, P., Roger, G.: Experimental realization of Einstein–Podolsky–Rosen–Bohm Gedankenexperiment: a new violation of Bell's Inequalities. Phys. Rev. Lett. **49**, 91–94 (1982)
40. Aspect, A., Dalibard, J., Roger, G.: Experimental test of Bell's Inequalities using time-varying analyzers. Phys. Rev. Lett. **49**, 1804–1807 (1982)
41. Kolmogorov, A.N.: Foundations of the Theory of Probability. Chelsea Publishing, New York (1956)
42. Accardi, L.: The probabilistic roots of the quantum mechanical paradoxes. In: Diner, S., Fargue, D., Lochak, G., Selleri, F. (eds.) The Wave-Particle Dualism: A Tribute to Louis de Broglie on his 90th Birthday, pp. 297–330. Springer, Dordrecht (1984)
43. Aerts, D.: A possible explanation for the probabilities of quantum mechanics and a macroscopical situation that violates Bell inequalities. In: Mittelstaedt, P., Stachow, E.W. (eds.): Recent Developments in Quantum Logic, Grundlagen der Exacten Naturwissenschaften, Wissenschaftverlag, vol. 6, pp. 235–251. Bibliographisches Institut, Mannheim (1985)
44. Aerts, D.: A possible explanation for the probabilities of quantum mechanics. J. Math. Phys. **27**, 202–210 (1986)
45. Aerts, D.: The origin of the non-classical character of the quantum probability model. In: Blanquiere, A., Diner, S., Lochak, G. (eds.) Information, Complexity, and Control in Quantum Physics, pp. 77–100. Springer, Wien-New York (1987)
46. Redhead, M.: Incompleteness, Nonlocality and Realism. Clarendon Press, Oxford (1987)

47. Pitowsky, I.: Quantum Probability, Quantum Logic. Lecture Notes in Physics, vol. 321. Springer, Heidelberg (1989)
48. Hampton, J.A.: Overextension of conjunctive concepts: Evidence for a unitary model for concept typicality and class inclusion. J. Exp. Psychol. Learn. Mem. Cogn. **14**, 12–32 (1988)
49. Hampton, J.A.: Inheritance of attributes in natural concept conjunctions. Mem. Cogn. **15**, 55–71 (1987)
50. Hampton, J.A.: The combination of prototype concepts. In: Schwanenugel, P. (ed.) The Psychology of Word Meanings. Erlbaum, Hillsdale (1991)
51. Storms, G., De Boeck, P., Van Mechelen, I., Geeraerts, D.: Dominance and non-commutativity effects in concept conjunctions: extensional or intensional basis? Mem. Cogn. **21**, 752–762 (1993)
52. Hampton, J.A.: Conjunctions of visually-based categories: overextension and compensation. J. Exp. Psychol. Learn. Mem. Cogn. **22**, 378–396 (1996)
53. Hampton, J.A.: Conceptual combination: conjunction and negation of natural concepts. Mem. Cogn. **25**, 888–909 (1997)
54. Storms, G., de Boeck, P., Hampton, J.A., van Mechelen, I.: Predicting conjunction typicalities by component typicalities. Psychon. Bull. Rev. **6**, 677–684 (1999)
55. Tversky, A., Kahneman, D.: Judgments of and by representativeness. In: Kahneman, D., Slovic, P., Tversky, A. (eds.): Judgment under Uncertainty: Heuristics and Biases. Cambridge University Press, Cambridge (1982)
56. Tversky, A., Kahneman, D.: Extension versus intuitive reasoning: the conjunction fallacy in probability judgment. Psychol. Rev. **90**, 293–315 (1983)
57. Hampton, J.A.: Disjunction of natural concepts. Mem. Cogn. **16**, 579–591 (1988)
58. Carlson, B.W., Yates, J.F.: Disjunction errors in qualitative likelihood judgment. Organ. Behav. Hum. Decis. Process. **44**, 368–379 (1989)
59. Tversky, A., Shafir, E.: The disjunction effect in choice under uncertainty. Psychol. Sci. **3**, 305–309 (1992)
60. Bar-Hillel, M., Neter, E.: How alike is it versus how likely is it: a disjunction fallacy in probability judgments. J. Personal. Soc. Psychol. **65**, 1119–1131 (1993)
61. Croson, R.T.A.: The disjunction effect and reason-based choice in games. Organ. Behav. Hum. Decis. Process. **80**, 118–133 (1999)
62. Kühberger, A., Komunska, D., Perner, J.: The disjunction effect: Does it exist for two-step gambles? Organ. Behav. Hum. Decis. Process. **85**, 250–264 (2001)
63. Li, S., Taplin, J.E.: Examining whether there is a disjunction effect in prisoner's dilemma games. Chin. J. Psychol. **44**, 25–46 (2002)
64. van Dijk, E., Zeelenberg, M.: The discounting of ambiguous information in economic decision making. J. Behav. Decis. Mak. **16**, 341–352 (2003)
65. van Dijk, E., Zeelenberg, M.: The dampening effect of uncertainty on positive and negative emotions. J. Behav. Decis. Mak. **19**, 171–176 (2006)
66. Bauer, M.I., Johnson-Laird, P.N.: How diagrams can improve reasoning. Psychol. Sci. **4**, 372–378 (2006)
67. Lambdin, C., Burdsal, C.: The disjunction effect reexamined: relevant methodological issues and the fallacy of unspecified percentage comparisons. Organ. Behav. Hum. Decis. Process. **103**, 268–276 (2007)
68. Bagassi, M., Macchi, L.: The 'vanishing' of the disjunction effect by sensible procrastination. Mind & Society **6**, 41–52 (2007)
69. Hristova, E., Grinberg, M.: Disjunction effect in prisoner's dilemma: Evidences from an eye-tracking study. In: Cogsci 2008, Proceedings, Washington, July 22–26 (2008)
70. Allais, M.: Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Américaine. Econometrica **21**, 503–546 (1953)
71. Ellsberg, D.: Risk, ambiguity, and the savage axioms. Q. J. Economics **75**, 643–669 (1961)
72. Savage, L.J.: The Foundations of Statistics. Wiley, New York (1954)
73. von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton (1944)
74. Busemeyer, J.R., Wang, Z., Townsend, J.T.: Quantum dynamics of human decision-making. J. Math. Psychol. **50**, 220–241 (2006)
75. Pothos, E.M., Busemeyer, J.R.: A quantum probability explanation for violations of 'rational' decision theory. Proc. R. Soc. B (2009)
76. Khrennikov, A., Haven, E.: Quantum mechanics and violations of the sure-thing principle: the use of probability interference and other concepts. J. Math. Psychol. **53**, 378–388 (2009)